

THE VIBRATIONAL FREQUENCIES OF THE ELASTIC BODY AND ITS GEOMETRIC QUANTITIES

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ABSTRACT. For a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, we explicitly calculate the first two coefficients of the asymptotic expansion of the trace of the strongly continuous semigroup associated with the Navier-Lamé operator on Ω as $t \rightarrow 0^+$. These coefficients (i.e., spectral invariants) provide precise information for the volume of the elastic body Ω and the surface area of the boundary $\partial\Omega$ in terms of the spectrum of the Navier-Lamé problem. As an application, we show that an n -dimensional ball is uniquely determined by its Navier-Lamé spectrum among all bounded elastic body with smooth boundary.

1. INTRODUCTION

For the Navier-Lamé equations, one of important problems is to study the geometry of the elastic body from the vibrational frequencies of the elastic body, because this just reveals the true behavior of the elastic body. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary $\partial\Omega$. Let P be the Navier-Lamé operator:

$$(1.1) \quad P\mathbf{u} := -\tau\Delta\mathbf{u} - (\tau + \mu)\nabla(\nabla \cdot \mathbf{u}), \quad \mathbf{u} = (u_1, u_2, \dots, u_n),$$

where ∇ is the gradient operator, Δ is the Laplacian, and τ and μ are Lamé parameters with $\tau > 0$, $\tau + \mu > 0$ (see [6], [14], [20], [33], [34]). We denote by P^- and P^+ the Navier-Lamé operators with the Dirichlet and Neumann boundary conditions, respectively. For the Derivation of the Navier-Lamé equation, its mechanical meaning and the explanation of the Dirichlet and Neumann boundary conditions, we refer the reader to [2] or [33].

According to theory of the elliptic equations, the Navier-Lamé operator P can generate analytic semigroup $U^-(t)$ (respectively, $U^+(t)$) with respect to the Dirichlet (respectively, Neumann) boundary condition in a suitable space of vector-valued functions (see [27], [28], [29], [31], [35]). Furthermore, there exists a function-valued matrix $\mathbf{K}^-(t, x, y)$ (respectively, $\mathbf{K}^+(t, x, y)$), which is called the heat kernel or fundamental solution, such that

$$(1.2) \quad (U^-(t)\mathbf{v}_0)(x) = \int_{\Omega} \mathbf{K}^-(t, x, y)\mathbf{v}_0(y)dy, \quad \mathbf{v}_0 \in [H_0^1(\Omega)]^n$$

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(respectively,

$$(1.3) \quad (U^+(t)\mathbf{v}_0)(x) = \int_{\Omega} \mathbf{K}^+(t, x, y)\mathbf{v}_0(y)dy, \quad \mathbf{v}_0 \in [H^1(\Omega)]^n.$$

Thus, $\mathbf{v}^-(t, x) := (U^-(t)\mathbf{v}_0)(x)$ and $\mathbf{v}^+(t, x) := (U^+(t)\mathbf{v}_0)(x)$ respectively satisfy the initial-boundary problems for the elastodynamic evolution equations

$$(1.4) \quad \begin{cases} \mathbf{v}_t^- - \tau \Delta \mathbf{v}^- - (\tau + \mu) \nabla(\nabla \cdot \mathbf{v}^-) = 0 & \text{in } (0, +\infty) \times \Omega, \\ \mathbf{v}^- = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ \mathbf{v}^-(0, x) = \mathbf{v}_0 & \text{on } \{0\} \times \Omega \end{cases}$$

and

$$(1.5) \quad \begin{cases} \mathbf{v}_t^+ - \tau \Delta \mathbf{v}^+ - (\tau + \mu) \nabla(\nabla \cdot \mathbf{v}^+) = 0 & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial \mathbf{v}^+}{\partial \nu} = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ \mathbf{v}^+(0, x) = \mathbf{v}_0 & \text{on } \{0\} \times \Omega. \end{cases}$$

On the other hand, since the Navier-Lamé operator P^- (respectively, P^+) is a unbounded self-adjoint positive operator in $[H_0^1(\Omega)]^n$ (respectively, $(H^1(\Omega))^n$) with discrete spectrum $0 < \lambda_1^- < \lambda_2^- \leq \dots \leq \lambda_k^- \leq \dots \rightarrow +\infty$ (respectively, $0 = \lambda_1^+ < \lambda_2^+ \leq \dots \leq \lambda_k^+ \leq \dots \rightarrow +\infty$), one has

$$(1.6) \quad P^{\mp} \mathbf{u}_k^{\mp} = \lambda_k^{\mp} \mathbf{u}_k^{\mp},$$

where $\mathbf{u}_k^- \in [H_0^1(\Omega)]^n$ (or $\mathbf{u}_k^+ \in [H^1(\Omega)]^n$) are the corresponding orthogonal eigenvectors. (1.6) can be rewritten as

$$(1.7) \quad \begin{cases} -\tau \Delta \mathbf{u}_k^- - (\tau + \mu) \nabla(\nabla \cdot \mathbf{u}_k^-) = \lambda_k^- \mathbf{u}_k^- & \text{in } \Omega, \\ \mathbf{u}_k^- = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$(1.8) \quad \begin{cases} -\tau \Delta \mathbf{u}_k^+ - (\tau + \mu) \nabla(\nabla \cdot \mathbf{u}_k^+) = \lambda_k^+ \mathbf{u}_k^+ & \text{in } \Omega, \\ \frac{\partial \mathbf{u}_k^+}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, the eigenvalue problem (1.7) and (1.8) can be immediately obtained by considering the solution of the form $\mathbf{v}^{\mp}(t, x) = T(t)\mathbf{u}^{\mp}(x)$ in Navier-Lamé evolution equations (1.4) and (1.5). The Navier-Lamé eigenvalues are physical quantities because they just are the vibrational frequencies of an elastic body in the two or three dimensions.

An interesting question, which is similar to the famous Kac question for the Dirichlet-Laplacian (see [17], [24] or [38]), is: “can one hear the shape of an elastic body by hearing the vibrational frequencies (or pitches) of elastic body?” More precisely, we have elastic body of different shapes. You hit them, and then you listen to the frequencies of elastic wave. Can you tell the shape (or the geometric quantities) of the elastic body?

In this paper, some surprising and interesting results are obtained by considering the Navier-Lamé operator semigroup $U^{\mp}(t) = e^{-tP^{\mp}}$ and by using some new methods of pseudo-differential operators. The following theorem is the main result of this paper:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary $\partial\Omega$, and let $0 < \lambda_1^- < \lambda_2^- \leq \dots \leq \lambda_k^- \leq \dots$ (respectively, $0 = \lambda_1^+ < \lambda_2^+ \leq \lambda_3^+ \leq \dots \leq \lambda_k^+ \leq \dots$) be the eigenvalues of the Navier-Lamé operator P^- (respectively, P^+) with respect to the Dirichlet*

(respectively, Neumann) boundary condition. Then

$$(1.9) \quad \sum_{k=1}^{\infty} e^{-\lambda_k^{\mp} t} = \text{Tr}(e^{-tP^{\mp}}) = \left[\frac{(n-1)}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} \right] |\Omega| \\ \mp \frac{1}{4} \left[\frac{(n-1)}{(4\pi\tau t)^{(n-1)/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{(n-1)/2}} \right] |\partial\Omega| + O(t^{1-\frac{n}{2}}) \quad \text{as } t \rightarrow 0^+.$$

Here $|\Omega|$ denotes the n -dimensional volume of Ω , and $|\partial\Omega|$ denotes the $(n-1)$ -dimensional volume of $\partial\Omega$.

Our result shows that not only the volume $|\Omega|$ but also the surface area $|\partial\Omega|$ can be known if we know all Navier-Lamé eigenvalues with respect to the Dirichlet (respectively, Neumann) boundary condition. Roughly speaking, one can “hear” the volumes of the domain and the surface area of its boundary $\partial\Omega$ by “hearing” all the pitches of the vibration of an elastic body.

The key ideas of this paper are as follows. If \mathbf{u}_k^{\mp} is the normalized eigenvector of Navier-Lamé problem with eigenvalue λ_k^{\mp} , the Navier-Lamé heat kernel $\mathbf{K}^{\mp}(t, x, y)$ is given by

$$(1.10) \quad \mathbf{K}^{\mp}(t, x, y) = \sum_{k=1}^{\infty} e^{-t\lambda_k^{\mp}} \mathbf{u}_k^{\mp}(x) \otimes \mathbf{u}_k^{\mp}(y).$$

Thus the integral of the trace of $\mathbf{K}^{\mp}(t, x, y)$ is actually a spectral invariants: by (1.10), we can compute

$$(1.11) \quad \text{Tr} \left(\int_{\Omega} \mathbf{K}^{\mp}(t, x, x) dx \right) = \sum_{k=1}^{\infty} e^{-t\lambda_k^{\mp}}.$$

To further analyze the geometric content of the spectrum, we calculate the same trace by an entirely different way: we constructs the heat kernel from P^{\pm} by the Cauchy integral formula:

$$e^{-tP^{\mp}} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\lambda I - P^{\mp})^{-1} d\lambda,$$

where \mathcal{C} is a suitable curve in the complex plane in the positive direction around the spectrum of P^{\mp} . By calculating the full symbols of pseudodifferential operator, and then applying technique of McKean-Singer, we explicitly computes the integral of the trace of $e^{-tP^{\mp}}$. We can show that the integral of the trace has an asymptotic expansion

$$(1.12) \quad \text{Tr} \left(\int_{\Omega} \mathbf{K}^{\mp}(t, x, x) dx \right) \sim a_0 t^{-n/2} + a_1^{\mp} t^{-(n-1)/2} + \dots \quad \text{as } t \rightarrow 0^+,$$

where $a_0 = \left[\frac{(n-1)}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} \right] |\Omega|$, $a_1^{\mp} = \mp \frac{1}{4} \left[\frac{(n-1)}{(4\pi\tau t)^{(n-1)/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{(n-1)/2}} \right] |\partial\Omega|$. More exactly, we can apply the Seeley’s calculus in the interior of Ω (see [32] or [12]) to the symbols of the Navier-Lamé operator to get the coefficient a_0 . However, the Seeley’s method can’t be used to deal with the boundary case for the Navier-Lamé operator (cf. [32] or [11]). This has become a stumbling block in the expansion of the heat trace for the corresponding elastic operator. To overcome this problem and to obtain the second coefficient a_1 , we will approximate the heat kernel near the boundary locally by the “method of images.” Locally, the boundary looks like the superplane $x_n = 0$ in the \mathbb{R}^n ; letting $x \rightarrow x^*$ be the reflection $(x_1, \dots, x_{n-1}, x_n) \rightarrow (x_1, \dots, x_{n-1}, -x_n)$, the kernel $\mathbf{K}^{\mp}(t, x, y) = \mathbf{K}(t, x, y) \mp \mathbf{K}(t, x, y^*)$ (or its normal derivative) vanishes on $x_n = 0$. By further estimating the traces of these two heat kernels, we finally obtain coefficient a_1^{\mp} .

As an application of theorem 1.1, we can prove the following spectral rigidity result:

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose that the Navier-Lamé spectrum with respect to the Dirichlet (respectively, Neumann) boundary condition, is equal to that of B_r , a ball of radius r . Then $\Omega = B_r$.*

Corollary 1.2 also shows that a ball is uniquely determined by its Navier-Lamé spectrum among all Euclidean bounded domains with smooth boundary.

2. SOME NOTATIONS AND A LEMMAS

If W is an open subset of \mathbb{R}^n , we denote by $S_{1,0}^m = S_{1,0}^m(W, \mathbb{R}^n)$ the set of all $p \in C^\infty(W, \mathbb{R}^n)$ such that for every compact set $O \subset W$ we have

$$(2.1) \quad |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{O,\alpha,\beta} (1 + |\xi|)^{m-|\alpha|}, \quad x \in O, \xi \in \mathbb{R}^n$$

for all $\alpha, \beta \in \mathbb{N}_+^n$. The elements of $S_{1,0}^m$ are called symbols (or full symbols) of order m . It is clear that $S_{1,0}^m$ is a Fréchet space with semi-norms given by the smallest constants which can be used in (2.1) (i.e.,

$$\|p\|_{O,\alpha,\beta} = \sup_{x \in O} |(D_x^\beta D_\xi^\alpha p(x, \xi)) (1 + |\xi|)^{|\alpha|-m}|.$$

Let $p(x, \xi) \in S_{1,0}^m$. A pseudo-differential operator in an open set $W \subset \mathbb{R}^n$ is essentially defined by a Fourier integral operator (cf. [18], [16], [37], [12]):

$$(2.2) \quad P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(x, \xi) e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi,$$

and denoted by OPS^m . Here $u \in C_0^\infty(W)$ and $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} u(y) dy$ is the Fourier transform of u . If there are smooth $p_{m-j}(x, \xi)$, homogeneous in ξ of degree $m-j$ for $|\xi| \geq 1$, that is, $p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)$ for $r, |\xi| \geq 1$, and if

$$(2.3) \quad p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$$

in the sense that

$$(2.4) \quad p(x, \xi) - \sum_{j=0}^l p_{m-j}(x, \xi) \in S_{1,0}^{m-l-1},$$

for all l , then we say $p(x, \xi) \in S_{cl}^m$, or just $p(x, \xi) \in S^m$. We call $p_m(x, \xi)$ the principal symbols of $P(x, D)$.

An operator P is said to be an elliptic pseudodifferential operator of order m if for every compact $O \subset \Omega$ there exists a positive constant $c = c(O)$ such that

$$|p(x, \xi)| \geq c|\xi|^m, \quad x \in O, |\xi| \geq 1$$

for any compact set $O \subset \Omega$. If P is a non-negative elliptic pseudodifferential operator of order m , then the spectrum of P lies in a right half-plane and has a finite lower bound $\tau(P) = \inf\{\operatorname{Re} \lambda \mid \lambda \in \sigma(P)\}$. We can modify $p_m(x, \xi)$ for small ξ such that $p_m(x, \xi)$ has a positive lower bound throughout and lies in $\{\lambda = re^{i\theta} \mid r > 0, |\theta| \leq \theta_0\}$, where $\theta_0 \in (0, \frac{\pi}{2})$.

According to [12], the resolvent $(P - \lambda)^{-1}$ exists and is holomorphic in λ on a neighborhood of a set

$$W_{r_0, \epsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq r_0, \arg \lambda \in [\theta_0 + \epsilon, 2\pi - \theta_0 - \epsilon], \operatorname{Re} \lambda \leq \tau(P) - \epsilon\}$$

(with $\epsilon > 0$). There exists a parametrix Q'_λ on a neighborhood of a possibly larger set (with $\delta > 0, \epsilon > 0$)

$$V_{\delta, \epsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq \delta \text{ or } \arg \lambda \in [\theta_0 + \epsilon, 2\pi - \theta_0 - \epsilon]\}$$

such that this parametrix coincides with $(P - \lambda)^{-1}$ on the intersection. Its symbol $q(x, \xi, \lambda)$ in local coordinates is holomorphic in λ there and has the form (cf. Section 3.3 of [12])

$$(2.5) \quad q(x, \xi, \lambda) \sim \sum_{l \geq 0} q_{-m-l}(x, \xi, \lambda)$$

where

$$(2.6) \quad \begin{aligned} q_{-m} &= (p_m(x, \xi) - \lambda)^{-1}, \quad q_{-m-1} = b_{1,1}(x, \xi) q_{-m}^2, \\ \dots, \quad q_{-m-l} &= \sum_{k=1}^{2l} b_{l,k}(x, \xi) q_{-m}^{k+1}, \dots, \end{aligned}$$

with symbols $b_{l,k}$ independent of λ and homogeneous of degree $mk - l$ in ξ for $|\xi| \geq 1$. The semigroup e^{-tP} can be defined from P by the Cauchy integral formula (see p. 4 of [10]):

$$e^{-tP} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\lambda I - P)^{-1} d\lambda,$$

where \mathcal{C} is a suitable curve in the complex plane in the positive direction around the spectrum of P .

Clearly, the Navier-Lamé operator $P\mathbf{u} = -\tau \Delta \mathbf{u} - (\tau + \mu) \nabla(\nabla \cdot \mathbf{u})$, $\mathbf{u} = (u_1, u_2, \dots, u_n)$, can be rewritten as $P\mathbf{u}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\mathbf{A}(x, \xi)) e^{i\langle x, \xi \rangle} \hat{\mathbf{u}}^T(\xi) d\xi$, where

$$(2.7) \quad \mathbf{A}(x, \xi) = \begin{pmatrix} \tau|\xi|^2 + (\tau + \mu)\xi_1^2 & (\tau + \mu)\xi_1\xi_2 & \dots & (\tau + \mu)\xi_1\xi_n \\ (\tau + \mu)\xi_2\xi_1 & \tau|\xi|^2 + (\tau + \mu)\xi_2^2 & \dots & (\tau + \mu)\xi_2\xi_n \\ \dots & \dots & \dots & \dots \\ (\tau + \mu)\xi_n\xi_1 & (\tau + \mu)\xi_n\xi_2 & \dots & \tau|\xi|^2 + (\tau + \mu)\xi_n^2 \end{pmatrix}$$

and $\hat{\mathbf{u}}^T = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$.

We will calculate the reverse of matrix $\lambda I - \mathbf{A}$ and further give its trace.

Lemma 2.1. *For any integers $n \geq 1$, we have*

$$(2.8) \quad \begin{aligned} \operatorname{Tr}((\lambda I - \mathbf{A})^{-1}) &= (\lambda - \tau|\xi|^2)^{-1} (\lambda - (2\tau + \mu)|\xi|^2)^{-1} \\ &\quad \times [n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2]. \end{aligned}$$

Proof. (i) When $n = 1$, it is clear that

$$(\lambda I - \mathbf{A})^{-1} = (\lambda I - (2\tau + \mu)|\xi|^2)^{-1}.$$

(ii) When $n = 2$, we have

$$\begin{aligned} (\lambda I - \mathbf{A})^{-1} &= (\lambda - \tau|\xi|^2)^{-1} (\lambda - (2\tau + \mu)|\xi|^2)^{-1} \\ &\quad \times \begin{pmatrix} \lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2 & (\tau + \mu)\xi_1\xi_2 \\ (\tau + \mu)\xi_2\xi_1 & \lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2 \end{pmatrix}, \end{aligned}$$

which implies

$$\text{Tr}((\lambda I - \mathbf{A})^{-1}) = (\lambda - \tau|\xi|^2)^{-1}(\lambda - (2\tau + \mu)|\xi|^2)^{-1}(2\lambda - (3\tau + \mu)|\xi|^2).$$

(iii) When $n = 3$, we get

$$\begin{aligned} (\lambda I - \mathbf{A})^{-1} &= (\lambda - \tau|\xi|^2)^{-2}(\lambda - (2\tau + \mu)|\xi|^2)^{-1} \cdot (\lambda - \tau|\xi|^2) \\ &\times \begin{pmatrix} \lambda - \tau|\xi|^2 - (\tau + \mu)(\xi_2^2 + \xi_3^2) & (\tau + \mu)\xi_1\xi_2 & (\tau + \mu)\xi_1\xi_3 \\ (\tau + \mu)\xi_2\xi_1 & \lambda - \tau|\xi|^2 - (\tau + \mu)(\xi_1^2 + \xi_3^2) & (\tau + \mu)\xi_2\xi_3 \\ (\tau + \mu)\xi_3\xi_1 & (\tau + \mu)\xi_3\xi_2 & \lambda - \tau|\xi|^2 - (\tau + \mu)(\xi_1^2 + \xi_2^2) \end{pmatrix} \end{aligned}$$

so that

$$\text{Tr}(\lambda I - \mathbf{A})^{-1} = (\lambda - \tau|\xi|^2)^{-1}(\lambda - (2\tau + \mu)|\xi|^2)^{-1}(3\lambda - (5\tau + 2\mu)|\xi|^2).$$

(iv) For $n = 4$, we find that

$$(\lambda I - \mathbf{A})^{-1} = (\lambda - \tau|\xi|^2)^{-3}(\lambda - (2\tau + \mu)|\xi|^2)^{-1} \begin{pmatrix} b_{11}^* & b_{12}^* & b_{13}^* & b_{14}^* \\ b_{21}^* & b_{22}^* & b_{23}^* & b_{24}^* \\ b_{31}^* & b_{32}^* & b_{33}^* & b_{34}^* \\ b_{41}^* & b_{42}^* & b_{43}^* & b_{44}^* \end{pmatrix},$$

where b_{ij}^* is the (i,j)-cofactor of the matrix $(\lambda - A)$. It is easy to check that

$$\begin{aligned} b_{11}^* &= (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_3^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_4^2) \\ &\quad - 2(\tau + \mu)^3\xi_2^2\xi_3^2\xi_4^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2)(\tau + \mu)^2\xi_3^2\xi_4^2 \\ &\quad - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_3^2)(\tau + \mu)^2\xi_2^2\xi_4^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_4^2)(\tau + \mu)^2\xi_2^2\xi_3^2, \\ b_{22}^* &= (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_3^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_4^2) \\ &\quad - 2(\tau + \mu)^3\xi_1^2\xi_3^2\xi_4^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2)(\tau + \mu)^2\xi_3^2\xi_4^2 \\ &\quad - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_3^2)(\tau + \mu)^2\xi_1^2\xi_4^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_4^2)(\tau + \mu)^2\xi_1^2\xi_3^2, \\ b_{33}^* &= (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_4^2) \\ &\quad - 2(\tau + \mu)^3\xi_1^2\xi_2^2\xi_4^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2)(\tau + \mu)^2\xi_2^2\xi_4^2 \\ &\quad - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2)(\tau + \mu)^2\xi_1^2\xi_4^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_4^2)(\tau + \mu)^2\xi_1^2\xi_2^2, \\ b_{44}^* &= (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2)(\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_3^2) \\ &\quad - 2(\tau + \mu)^3\xi_1^2\xi_2^2\xi_3^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_1^2)(\tau + \mu)^2\xi_2^2\xi_3^2 \\ &\quad - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_2^2)(\tau + \mu)^2\xi_1^2\xi_3^2 - (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_3^2)(\tau + \mu)^2\xi_1^2\xi_2^2. \end{aligned}$$

A direct calculation shows that

$$b_{11}^* + b_{22}^* + b_{33}^* + b_{44}^* = (\lambda - \tau|\xi|^2)^2(4\lambda - (7\tau + 3\mu)|\xi|^2).$$

Thus, we have that

$$\begin{aligned} \text{Tr}((\lambda I - \mathbf{A})^{-1}) &= (\lambda - \tau|\xi|^2)^{-3}(\lambda - (2\tau + \mu)|\xi|^2)^{-1}(\lambda - \tau|\xi|^2)^2(4\lambda - (7\tau + 3\mu)|\xi|^2) \\ &= (\lambda - \tau|\xi|^2)^{-1}(\lambda - (\tau + \mu)|\xi|^2)^{-1}(4\lambda - (7\tau + 3\mu)|\xi|^2). \end{aligned}$$

(v) For general integer $n \geq 1$, since

$$(\lambda \mathbf{I} - \mathbf{A})^{-1} = |\lambda \mathbf{I} - \mathbf{A}|^{-1} \begin{pmatrix} c_{11}^* & c_{12}^* & \cdots & c_{1n}^* \\ c_{21}^* & c_{22}^* & \cdots & c_{2n}^* \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1}^* & c_{n2}^* & \cdots & c_{nn}^* \end{pmatrix},$$

where c_{ij}^* is the (i,j) -cofactor of the matrix $(\lambda \mathbf{I} - \mathbf{A})$. It follows from the technique of (iv) that for any integers $n \geq 1$,

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda - \tau|\xi|^2)^{(n-1)} (\lambda - (2\tau + \mu)|\xi|^2)$$

and

$$\begin{aligned} c_{ii}^* &= \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_j^2) - \sum_{k=0}^{n-3} (n-k-2) (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_{l_1}^2) \\ &\quad \times \cdots \times (\lambda - \tau|\xi|^2 - (\tau + \mu)\xi_{l_k}^2) (\tau + \mu)^{n-k-1} \xi_{m_1}^2 \cdots \xi_{m_{n-k-1}}^2, \quad i = 1, 2, \dots, n, \end{aligned}$$

where

$$\begin{aligned} l_1, \dots, l_k &\in \{1, 2, \dots, n\} \setminus \{i\}, \quad l_1 < \cdots < l_k, \quad \text{and} \\ m_1, \dots, m_{n-k-1} &\in \{1, 2, \dots, n\} \setminus \{i, l_1, \dots, l_k\}, \quad m_1 < \cdots < m_{n-k-1}. \end{aligned}$$

It follows that

$$\sum_{i=1}^n c_{ii}^* = (\lambda - \tau|\xi|^2)^{n-2} [n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2].$$

Hence

$$\begin{aligned} \text{Tr}((\lambda \mathbf{I} - \mathbf{A})^{-1}) &= (\lambda - \tau|\xi|^2)^{-(n-1)} (\lambda - (2\tau + \mu)|\xi|^2)^{-1} (\lambda - \tau|\xi|^2)^{(n-2)} \\ &\quad \times [n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2] \\ &= (\lambda - \tau|\xi|^2)^{-1} (\lambda - (2\tau + \mu)|\xi|^2)^{-1} [n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2]. \end{aligned}$$

□

Now, we construct a pseudodifferential operator B to approximate the resolvent $(\lambda I - P)^{-1}$ as follows: let

$$(2.9) \quad \mathbf{b}(x, \xi, \lambda) \sim \mathbf{b}_0(x, \xi, \lambda) + \cdots + \mathbf{b}_m(x, \xi, \lambda) + \cdots$$

be the expansion of the full symbol of B . Suppose that the complex parameter λ have homogeneity 2 (cf. [32] or [9]). Let the \mathbf{b}_m be homogeneous of order $-2 - m$ in the variables (ξ, λ) . This infinite sum defines \mathbf{b} asymptotically. Our purpose is to define \mathbf{b} so that

$$(2.10) \quad \sigma(B(\lambda I - P)) \sim I,$$

where $\sigma(T)$ denotes the full symbol of pseudodifferential operator T .

We have the following:

Lemma 2.2. *Let the full symbol \mathbf{b} of B have the form (2.9) and let B satisfy (2.10) (i.e.,*

$$(2.11) \quad \sum_{\alpha \geq 0} (d_\xi^\alpha \mathbf{b}) \cdot (D_x^\alpha (\sigma(\lambda I - P))) / \alpha! \sim I).$$

Then $\mathbf{b}_0(x, \xi, \lambda) = (\lambda \mathbf{I} - \mathbf{A})^{-1}$ and $\mathbf{b}_m(x, \xi, \lambda) = 0$ for all $m \geq 1$.

Proof. Denote $\mathbf{a}_2 = (\lambda I - \mathbf{A})$, $\mathbf{a}_1 = 0$ and $\mathbf{a}_0 = 0$. We decompose sum (2.10) into orders of homogeneity (see [15], [16] or p. 13 of [37]):

$$\sigma(B(\lambda I - P)) \sim \sum_{m=0}^{\infty} \left(\sum_{m=j+|\alpha|+2-k} (d_{\xi}^{\alpha} \mathbf{b}_j) \cdot (D_x^{\alpha} \mathbf{a}_k) / \alpha! \right).$$

The sum is over terms which are homogeneous of order $-m$. Thus (2.10) leads to the following equations

$$\begin{aligned} I &= \sum_{0=j+|\alpha|+2-k} (d_{\xi}^{\alpha} \mathbf{b}_j) (D_x^{\alpha} \mathbf{a}_k) / \alpha! = \mathbf{b}_0 \mathbf{a}_2, \\ 0 &= \sum_{m=j+|\alpha|+2-k} (d_{\xi}^{\alpha} \mathbf{b}_j) (D_x^{\alpha} \mathbf{a}_k) / \alpha! \\ &= \mathbf{b}_m \mathbf{a}_2 + \sum_{\substack{m=j+|\alpha|+2-k \\ j < m}} (d_{\xi}^{\alpha} \mathbf{b}_j) (D_x^{\alpha} \mathbf{a}_k) / \alpha!, \quad m \geq 1. \end{aligned}$$

These equations determine the \mathbf{b}_m inductively. In other words, we have

$$\begin{aligned} \mathbf{b}_0 &= \mathbf{a}_2^{-1}, \quad \mathbf{b}_m = -\mathbf{b}_0 \left(\sum_{j < m} (d_{\xi}^{\alpha} \mathbf{b}_j) (D_x^{\alpha} \mathbf{a}_k) / \alpha! \right), \\ &\quad \text{for } m = j + |\alpha| + 2 - k. \end{aligned}$$

Since $\mathbf{a}_1 = \mathbf{a}_0 = 0$ and since \mathbf{a}_2 is independent of x , it can be immediately see that $\mathbf{b}_m(x, \xi, \lambda) = 0$ for all $m \geq 1$. \square

3. ASYMPTOTIC EXPANSION

Proof of Theorem 1.1. We calculate the asymptotic expansion of the trace of semigroup e^{-tP} as $t \rightarrow 0^+$. Note that the interior asymptotics are independent of the boundary condition; however, the boundary asymptotics depend on the Dirichlet (or Neumann) boundary conditions. It follows from Lemmas 2.1 and 2.2 that the full symbols of the operator $(\lambda I - P)^{-1}$ is $(\lambda \mathbf{I} - \mathbf{A})^{-1}$. In view of

$$e^{-tP} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\lambda I - P)^{-1} d\lambda,$$

where \mathcal{C} is a suitable curve in the complex plane in the positive direction around the spectrum of P , we find that

$$(3.1) \quad \mathbf{K}(t, x, y) = e^{-tP} \delta(x - y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi \rangle} \left[\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\lambda \mathbf{I} - \mathbf{A})^{-1} d\lambda \right] d\xi.$$

It follows from Lemma 2.1 that

$$\begin{aligned} \text{Tr}((\lambda I - \mathbf{A})^{-1}) &= (\lambda - \tau|\xi|^2)^{-1} (\lambda - (2\tau + \mu)|\xi|^2)^{-1} \\ &\quad \times [n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2], \quad \text{for } n = 1, 2, 3, \dots \end{aligned}$$

Thus, for any $W \subset \Omega$.

$$\begin{aligned}
 (3.2) \quad \text{Tr}\left(e^{-tP}|_W\right) &= \int_W \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x-x, \xi \rangle} \left[\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\text{Tr}((\lambda I - A)^{-1})) d\lambda \right] d\xi \right\} dx \\
 &= \int_W \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\lambda - \tau|\xi|^2)^{-1} (\lambda - (2\tau + \mu)|\xi|^2)^{-1} \right. \right. \\
 &\quad \left. \left. \times \left(n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2 \right) d\lambda \right] d\xi \right\} dx.
 \end{aligned}$$

Applying the residue theorem [1], we obtain

$$\begin{aligned}
 (3.3) \quad & \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} \frac{(n\lambda - ((2n-1)\tau + (n-1)\mu)|\xi|^2)}{(\lambda - \tau|\xi|^2)(\lambda - (2\tau + \mu)|\xi|^2)} d\lambda \\
 &= \frac{n\tau|\xi|^2 - ((2n-1)\tau + (n-1)\mu)|\xi|^2}{\tau|\xi|^2 - (2\tau + \mu)|\xi|^2} e^{-t\tau|\xi|^2} \\
 &+ \frac{n(2\tau + \mu)|\xi|^2 - ((2n-1)\tau + (n-1)\mu)|\xi|^2}{(2\tau + \mu)|\xi|^2 - \tau|\xi|^2} e^{-t(2\tau + \mu)|\xi|^2} \\
 &= (n-1)e^{-t\tau|\xi|^2} + e^{-t(2\tau + \mu)|\xi|^2}.
 \end{aligned}$$

From (3.2) and (3.3), we get

$$\begin{aligned}
 (3.4) \quad \text{Tr}\left(e^{-tP}|_W\right) &= \int_W \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[(n-1)e^{-t\tau|\xi|^2} + e^{-t(2\tau + \mu)|\xi|^2} \right] d\xi \right\} dx \\
 &= \int_W \left[\frac{n-1}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau + \mu)t)^{n/2}} \right] dx \\
 &= \left[\frac{n-1}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau + \mu)t)^{n/2}} \right] |W|.
 \end{aligned}$$

More importantly, (3.4) is still valid for any n -dimensional normal coordinate patch W covering a patch of $W \cap \partial\Omega$.

It remains to consider the boundary asymptotics. Let $x = (x'; x_n)$ be local coordinates for Ω near $\partial\Omega$. If \mathfrak{E} is a local frame on $\partial\Omega$; extend \mathfrak{E} to an n -dimensional local frame in a neighborhood of $\partial\Omega$ by parallel transport along the geodesic normal rays (see, p. 1101 of [21]). We will apply an “imagine method”, which stems from McKean-Singer in §5 of [25], to deal with the case of the boundary. Let $\mathcal{M} = \Omega \cup (\partial\Omega) \cup \Omega^*$ be the (closed) double of Ω , and Q the double to \mathcal{M} of the operator P . Define Q^- and Q^+ to be $Q|_{C^\infty(\bar{\Omega})}$ subject to $\mathbf{u} = 0$ and $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial\Omega$, respectively. The $\frac{\partial \mathbf{u}}{\partial t} = Q\mathbf{u}$ still has a nice fundamental solution $\mathbf{K}(t, x, y)$ of class $C^\infty[(0, \infty) \times (\mathcal{M} \setminus \partial\Omega)^2] \cap C^1(\mathcal{M}^2)$, approximable even on $\partial\Omega$, and the fundamental solution $\mathbf{K}^\mp(t, x, y)$ of $\frac{\partial \mathbf{u}}{\partial t} = Q^\mp \mathbf{u}$ can be expressed on $(0, \infty) \times \Omega \times \Omega$ as

$$(3.5) \quad \mathbf{K}^\mp(t, x, y) = \mathbf{K}(t, x, y) \mp \mathbf{K}(t, x, y^*),$$

y^* being the double of $y \in \Omega$ (see, p. 53 of [25]). We pick a self-double patch W of \mathcal{M} covering a patch $W \cap \partial\Omega$ endowed (see the diagram on p. 53 of [25]) with local coordinates x such that $\epsilon > x_n > 0$ in $W \cap \Omega$; $x_n = 0$ on $W \cap \partial\Omega$; $x_n(x^*) = -x_n(x)$; and the positive x_n -direction is

perpendicular to $\partial\Omega$. This products the following effect that

$$\begin{aligned}\delta_{jk}(\overset{*}{x}) &= -\delta_{jk}(x) \quad \text{for } j < k = n \text{ or } k < j = n, \\ &= \delta_{jk}(x) \quad \text{for } j, k < n \text{ or } j = k = n, \\ \delta_{jk}(x) &= 0 \quad \text{for } j < k = n \text{ or } k < j = n \text{ on } \partial\Omega.\end{aligned}$$

For any $W \subset \Omega$, by the previous technique we see that (3.4) still holds. For any n -dimensional normal coordinate patch W covering a patch $W \cap \partial\Omega$, it follows from (3.4) that

$$\begin{aligned}(3.6) \quad \int_{W \cap \Omega} \text{Tr}(\mathbf{K}(t, x, x)) dx &= \frac{1}{(4\pi\mu t)^{\frac{n}{2}}} \left[\int_{W \cap \Omega} \left[\frac{n-1}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} \right] dx \right. \\ &= \left. \left[\frac{n-1}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} \right] |W \cap \Omega| \quad \text{for all } t \geq 0.\end{aligned}$$

Next, for any small n -dimensional normal coordinate patch W covering a patch of $W \cap \partial\Omega$, noting that $|x - \overset{*}{x}| = x_n - (-x_n) = 2x_n$ we find by the method of pseudodifferential operator that

$$\begin{aligned}\int_{W \cap \Omega} \text{Tr}(\mathbf{K}(t, x, \overset{*}{x})) dx &= \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} e^{i\langle x - \overset{*}{x}, \xi \rangle} \\ &\quad \times \left[\frac{1}{2\pi i} \int_{\mathbb{C}} e^{-t\lambda} (\text{Tr}((\lambda I - A)^{-1})) d\lambda \right] d\xi \\ &= \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} e^{i\langle x - \overset{*}{x}, \xi \rangle} \left[(n-1)e^{-t\tau|\xi|^2} + e^{-t(2\tau+\mu)|\xi|^2} \right] d\xi \\ &= \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n\xi_n} \left\{ \int_{\mathbb{R}^{n-1}} e^{\langle 0, \xi' \rangle} \left[(n-1)e^{-t\tau(|\xi'|^2 + \xi_n^2)} \right. \right. \\ &\quad \left. \left. + e^{-t(2\tau+\mu)(|\xi'|^2 + x_n^2)} \right] d\xi' \right\} d\xi_n \\ &= \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n\xi_n} e^{-t\tau\xi_n^2} \left\{ \int_{\mathbb{R}^{n-1}} \left[(n-1)e^{-t\tau \sum_{j=1}^{n-1} \xi_j^2} \right] d\xi' \right\} d\xi_n \\ &\quad + \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n\xi_n} e^{-t(2\tau+\mu)\xi_n^2} \left\{ \int_{\mathbb{R}^{n-1}} \left[e^{-t(2\tau+\mu) \sum_{j=1}^{n-1} \xi_j^2} \right] d\xi' \right\} d\xi_n \\ &= \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{n-1}{(4\pi\tau t)^{n/2}} e^{-\frac{(2x_n)^2}{4\tau t}} dx' + \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} e^{-\frac{(2x_n)^2}{4(2\tau+\mu)t}} dx' \\ &= \int_0^\infty dx_n \int_{W \cap \partial\Omega} \frac{n-1}{(4\pi\tau t)^{n/2}} e^{-\frac{(2x_n)^2}{4\tau t}} dx' - \int_\epsilon^\infty dx_n \int_{W \cap \partial\Omega} \frac{n-1}{(4\pi\tau t)^{n/2}} e^{-\frac{(2x_n)^2}{4\tau t}} dx' \\ &\quad + \int_0^\infty dx_n \int_{W \cap \partial\Omega} \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} e^{-\frac{(2x_n)^2}{4(2\tau+\mu)t}} dx' \\ &\quad - \int_\epsilon^\infty dx_n \int_{W \cap \partial\Omega} \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} e^{-\frac{(2x_n)^2}{4(2\tau+\mu)t}} dx' \\ &= \frac{n-1}{4} \cdot \frac{|W \cap \partial\Omega|}{(4\pi\tau t)^{(n-1)/2}} + \frac{1}{4} \cdot \frac{|W \cap \partial\Omega|}{(4\pi(2\tau+\mu)t)^{(n-1)/2}} \\ &\quad - \int_{W \cap \partial\Omega} \left\{ \int_\epsilon^\infty \left[\frac{n-1}{(4\pi\tau t)^{n/2}} e^{-\frac{(2x_n)^2}{4\tau t}} + \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} e^{-\frac{(2x_n)^2}{4(2\tau+\mu)t}} \right] dx_n \right\} dx',\end{aligned}$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, and $\epsilon > 0$ is some fixed real number. It is easy to verify that for any fixed $\epsilon > 0$ and any integer $m \geq 1$,

$$\begin{aligned} \int_{\epsilon}^{\infty} \frac{1}{(4\pi\mu t)^{\frac{n}{2}}} e^{-\frac{(2x_n)^2}{4\tau t}} dx_n &= o(t^{m-\frac{n}{2}}) \quad \text{as } t \rightarrow 0^+, \\ \int_{\epsilon}^{\infty} \frac{1}{(4\pi(2\tau + \mu)t)^{\frac{n}{2}}} e^{-\frac{(2x_n)^2}{4(2\tau + \mu)t}} dx_n &= o(t^{m-\frac{n}{2}}) \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

Combining these we see that for any integer $m \geq 1$,

$$(3.7) \quad \int_{W \cap \Omega} \text{Tr}(\mathbf{K}(t, x, x^*)) dx = \frac{n-1}{4} \cdot \frac{|W \cap \partial\Omega|}{(4\pi\tau t)^{(n-1)/2}} + \frac{1}{4} \cdot \frac{|W \cap \partial\Omega|}{(4\pi(2\tau + \mu)t)^{(n-1)/2}} + o(t^{m-\frac{n}{2}}) \quad \text{as } t \rightarrow 0^+.$$

It follows from (3.4), (3.5), (3.6) and (3.7) we obtain

$$(3.8) \quad \begin{aligned} \int_{W \cap \Omega} \text{Tr}(\mathbf{K}^{\mp}(t, x, x)) dx &= \int_{W \cap \Omega} \text{Tr}(\mathbf{K}(t, x, x)) dx \\ &\mp \int_{W \cap \Omega} \text{Tr}(\mathbf{K}(t, x, x^*)) dx = \left[\frac{n-1}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau + \mu)t)^{n/2}} \right] |W \cap \Omega| \\ &\mp \frac{1}{4} \left[(n-1) \frac{|W \cap \partial\Omega|}{(4\pi\tau t)^{(n-1)/2}} + \frac{|W \cap \partial\Omega|}{(4\pi(2\tau + \mu)t)^{(n-1)/2}} \right] \\ &+ O(t^{m-\frac{n}{2}}) \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

and hence (1.9) holds. \square

Remark 3.1. Unlike the Laplacian on Riemannian manifold or other elliptic operators (see, for example, [26], [5], [9], [25] or [23]), in the expansion of the heat trace for the Navier-Lamé operator, there exist only two spectral invariants (i.e., volume and surface area). This originates from the following two facts: (i) the Lamé coefficients τ and μ are constants (so that the Navier-Lamé operator is an elliptic operator of constant coefficients in \mathbb{R}^n); (ii) the expansion (1.9) is an exact formula except for an additional term $O(t^{\frac{n-1}{2}} e^{-\frac{c}{\sqrt{t}}})$, which is exponential decay as $t \rightarrow 0^+$.

Now, we use the heat invariants of the Navier-Lamé spectrum which have been obtained from Theorem 1.1 to finish the proof of Corollary 1.2.

Proof of Corollary 1.2. By Theorem 1.1, we know that the first two coefficients $|\Omega|$ and $|\partial\Omega|$ of the asymptotic expansion in (1.9) are Navier-Lamé spectral invariants, i.e., $|\Omega| = |B_r|$ and $|\partial\Omega| = |\partial B_r|$. Thus $\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} = \frac{|\partial B_r|}{|B_r|^{(n-1)/n}}$. Note that for any $r > 0$, $\frac{|\partial B_r|}{|B_r|^{(n-1)/n}} = \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}$. According to the classical isometric inequality (which states that for any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, the following inequality holds:

$$\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} \geq \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}.$$

Moreover, equality obtains if and only if Ω is a ball, see [4] or p.183 of [5]), we immediately get $\Omega = B_r$. \square

Remark 3.2. By applying the Tauberian theorem (see, for example, Theorem 15.3 of p.30 of [19] or p.446 of [7]) for the first term on the right side of (1.9) (i.e., $\sum_{k=1}^{\infty} e^{-t\lambda_k^{\mp}} =$

$\int_0^\infty e^{-t\eta} dN(\eta) = \left[\frac{n-1}{(4\pi\tau t)^{n/2}} + \frac{1}{(4\pi(2\tau+\mu)t)^{n/2}} \right] |\Omega| + o(t^{n/2})$ as $t \rightarrow 0^+$, we can easily obtain the Weyl-type law for the Navier-Lamé eigenvalues:

$$(3.9) \quad N(\eta) = \max\{k \mid \lambda_k^\mp \leq \eta\} = \frac{|\Omega|}{\Gamma(\frac{n}{2} + 1)} \left[\frac{n-1}{(4\pi\tau)^{n/2}} + \frac{1}{(4\pi(2\tau+\mu))^{n/2}} \right] \eta^{\frac{n}{2}} + o(\eta^{\frac{n}{2}}), \quad \text{as } \eta \rightarrow +\infty.$$

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